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Hans汉斯

The Initial Boundary Value Problems for a Class of *n*-Dimensional Nonlinear Evolution Systems of Higher Order

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Abstract

In this paper, we study a class of *n*-dimensional nonlinear wave system of high order by using the Galerkin method, Sobolev embedding theorem and prior estimate of solution, and prove the existence and uniqueness of global strong solution to the initial boundary value problem.

Keywords

Nonlinear Wave System, Prior Estimate, Galerkin Method, Sobolev Embedding Theorem, The Global Strong Solution

一类高阶*n*维非线性发展方程组的初边值问题

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摘　要

本文研究了一类高阶*n*维非线性波动方程组，通过解的先验估计，利用Galerkin方法和Sobolev嵌入定理，

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证明了初边值问题整体强解的存在性和唯一性。

关键词

非线性波动方程组，先验估计，Galerkin方法，Sobolev嵌入定理，整体强解

1. 引言

在本文中，$\boldsymbol{u}\cdot\boldsymbol{v}=\sum_{i=1}^{N}u_i v_i$，$(\boldsymbol{u},\boldsymbol{v})=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}x$，$|\boldsymbol{u}|_{L^2(\Omega)}^2=(\boldsymbol{u},\boldsymbol{u})$，

$$[\boldsymbol{u},\boldsymbol{v}]=\int_0^t(\boldsymbol{u},\boldsymbol{v})\mathrm{d}t,\quad \|\boldsymbol{u}\|_{L^2(\Omega)}^2=[\boldsymbol{u},\boldsymbol{u}],\quad |\boldsymbol{u}|=\left(\sum_{i=1}^{N}u_i^2\right)^{\frac{1}{2}},$$

$$\boldsymbol{u}=\boldsymbol{u}(x,t)=(u_1,u_2,\cdots,u_N)^{\mathrm{T}},\quad \boldsymbol{v}=\boldsymbol{v}(x,t)=(v_1,v_2,\cdots,v_N)^{\mathrm{T}},$$

$$\boldsymbol{f}(\boldsymbol{u}_t)=\left(f_1(\boldsymbol{u}_t),f_2(\boldsymbol{u}_t),\cdots,f_N(\boldsymbol{u}_t)\right)^{\mathrm{T}},\quad \boldsymbol{\varphi}(x)=\left(\varphi_1(x),\varphi_2(x),\cdots,\varphi_N(x)\right)^{\mathrm{T}},$$

$$\boldsymbol{\psi}(x)=\left(\psi_1(x),\psi_2(x),\cdots,\psi_N(x)\right)^{\mathrm{T}}$$

其中 $\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，$\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，“T”表示转置。

有三种因素影响弹性杆内波的传播：非线性、弥散以及耗散。文[1]研究了在上述三种因素影响下的细长弹性杆中纵向应变波的传播问题。

文[2]在此基础上研究了一类四阶非线性耗散色散波动方程的初边值问题，非线性耗散色散波动方程组关于空间变量导数为 2 阶的情形在文[3]中有所研究，本文主要在此基础上研究了高阶的情形，关于高阶 n 维非线性耗散色散波动方程组的研究在已有文献中还未见到。

本文主要研究了下述在粘弹性力学中具有实际背景的一类高阶 n 维非线性波动方程组：

$$\begin{cases}\boldsymbol{u}_{tt}+(-1)^M\Delta^M\boldsymbol{u}+(-1)^M\Delta^M\boldsymbol{u}_t+(-1)^M\Delta^M\boldsymbol{u}_{tt}=\boldsymbol{f}(\boldsymbol{u}_t),\ (x,t)\in\Omega\times[0,T], & (1.1)\\ \boldsymbol{u}|_{t=0}=\varphi(x),\quad \boldsymbol{u}_t|_{t=0}=\psi(x), & (1.2)\\ D^\gamma\boldsymbol{u}|_{\partial\Omega\times[0,T]}=0,\quad 0\le|\gamma|\le M-1 & (1.3)\end{cases}$$

其中 Ω 为 R^n 中具有光滑边界的有界区域，$T>0$，M 为正整数，$\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ 为多重指标，$\gamma_i(i=1,2,\cdots,n)$ 为非负整数，且 $|\gamma|=\gamma_1+\gamma_2+\cdots+\gamma_n$。$\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega),\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，假设 $\boldsymbol{f}$ 满足：

$$\boldsymbol{f}(\boldsymbol{0})=\boldsymbol{0},\quad \boldsymbol{f}\in C^1 \tag{1.4}$$

且 Jacobian 矩阵 $\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}$ 半有界，即 $\exists k_0>0,\forall\xi\in R^n$ 满足：

$$\left(\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}\xi,\xi\right)\le k_0(\xi,\xi), \tag{1.5}$$

$$|\boldsymbol{f}(\boldsymbol{u})|\le a_1+b_1|\boldsymbol{u}|^{\frac{p}{2}}\ (a_1,b_1\text{为正常数}), \tag{1.6}$$

当 $2M<n$时，$2\le p<\frac{2n}{n-2M}$；当 $2M\ge n$时, $2\le p<+\infty$。

2. 主要结论

定义：$\boldsymbol{u}=\boldsymbol{u}(x,t)$ $\left((x,t)\in\Omega\times[0,T],\Omega\subset R^n\right)$称为问题(1.1)~(1.3)在$\Omega\times[0,T]$上的整体强解，若

$$\boldsymbol{u}\in L^\infty\left(0,T;H^{2M}(\Omega)\cap H_0^M(\Omega)\right),$$
$$\boldsymbol{u}_t\in L^\infty\left(0,T;H^{2M}(\Omega)\cap H_0^M(\Omega)\right),$$
$$\boldsymbol{u}_{tt}\in L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H_0^M(\Omega)\right),$$

对一切$\boldsymbol{d}(x,t)=\left(d_1(x,t),d_2(x,t),\cdots,d_N(x,t)\right)\in C\left([0,T];L^2(\Omega)\right)$成立

$$\int_0^T\left(\boldsymbol{u}_{tt}+(-1)^M\Delta^M\boldsymbol{u}+(-1)^M\Delta^M\boldsymbol{u}_t+(-1)^M\Delta^M\boldsymbol{u}_{tt}-\boldsymbol{f}(\boldsymbol{u}_t),\boldsymbol{d}(x,t)\right)\mathrm{d}t=0$$

且

$$\boldsymbol{u}|_{t=0}=\boldsymbol{\varphi}(x),\ \boldsymbol{u}_t|_{t=0}=\boldsymbol{\psi}(x)$$

定理 1：若$\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega),\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，条件(1.5)~(1.6)成立，则问题(1.1)~(1.3)存在上述意义下的整体强解$\boldsymbol{u}=\boldsymbol{u}(x,t)$。

定理 2：若$\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega),\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，条件(1.5)~(1.6)成立，则非线性问题(1.1)~(1.3)的整体强解唯一。

3. 主要结论的证明

首先介绍定理 1 的证明思想，即第一步构造问题(1.1)~(1.3)的近似解，然后对近似解作出先验估计，在作出先验估计基础上，由列紧性原理可证得问题(1.1)~(1.3)的整体强解。下面进行定理 1 的证明：

设$\left\{w_j(x)\middle|j=1,2,\cdots\right\}$为问题$\begin{cases}(-1)^M\Delta^M w_j=\lambda_j w_j\\ D^\gamma w_j(x)|_{\partial\Omega}=0 \quad 0\le|\gamma|\le M-1\end{cases}$的特征函数系，则由文[4]中的引理 1.1 有结论：$\left\{w_j(x)\middle|j=1,2,\cdots\right\}$分别构成$L^2(\Omega)$，$H_0^M(\Omega),H^{2M}\cap H_0^M(\Omega)$的正交基底。

构造问题(1.1)~(1.3)的近似解

$$u_{mi}=u_{mi}(x,t)=\sum_{j=1}^m a_{mij}(t)w_j(x), \tag{2.1}$$

并满足初值条件

$$u_{mi}(x,0)=\varphi_{mi}(x)=\sum_{j=1}^m a_{mij}w_j(x)\ \left(a_{mij}\text{将在下面选取}\right), \tag{2.2}$$

$$u_{mit}(x,0)=\psi_{mi}(x)=\sum_{j=1}^m b_{mij}w_j(x)\ \left(b_{mij}\text{将在下面选取}\right). \tag{2.3}$$

由 Galerkin 方法，该近似解应满足如下非线性常微分方程组的初值问题

$$(*)\ \begin{cases}(u_{mitt},w_k)+\left((-1)^M\Delta^M u_{mi},w_k\right)+\left((-1)^M\Delta^M u_{mit},w_k\right)+\left((-1)^M\Delta^M u_{mitt},w_k\right)=\left(f_i(\boldsymbol{u}_{mt}),w_k\right),0\le t\le T\ ,(2.4)\\ u_{mi}(x,0)=\varphi_{mi}(x),\\ u_{mit}(x,0)=\psi_{mi}(x),\\ i=1,2,\cdots,N;k=1,2,\cdots m.\end{cases}$$

由于$\boldsymbol{\varphi}(x)=\left(\varphi_1(x),\varphi_2(x),\cdots,\varphi_N(x)\right)^{\mathrm{T}}\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，

$\boldsymbol{\psi}(x)=\left(\psi_1(x),\psi_2(x),\cdots,\psi_N(x)\right)^{\mathrm{T}}\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，$\left\{w_j(x)\middle|j=1,2,\cdots\right\}$ 构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 的一组正交基，可选取适当的 a_{mij}，b_{mij} $(i=1,2,\cdots,N;j=1,2,3,\cdots)$ 使得当 $m\to+\infty$ 时，在空间 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中成立强收敛，$\varphi_{mi}(x)\to\varphi_i(x)$，$\psi_{mi}(x)\to\psi_i(x)$，$(i=1,2,\cdots,N)$。因假设 $\boldsymbol{f}\in C^1$，由非线性常微分方程组理论中 Peano 存在性定理知问题 $(*)$ 存在局部解 $u_{mi}(x,t)(i=1,2,3,\cdots,N)$，为得到问题 $(*)$ 在 $[0,T]$ 上的整体解，需证明问题 $(*)$ 中方程的右端 $\left(f_i(\boldsymbol{u}_{mt}),w_k\right)$ 的绝对值能用一与 $\boldsymbol{u}_m(x,t)$ 无关的正常数控制住，为此作出近似解 $u_{mi}(x,t)(i=1,2,3,\cdots,N)$ 的先验估计如下：

引理 3.1：设 $\boldsymbol{\varphi}(x)\in H_0^M(\Omega),\boldsymbol{\psi}(x)\in H_0^M(\Omega)$，选取 a_{mij},b_{mij}，使当 $m\to+\infty$ 时，$\boldsymbol{u}_m(x,0)\to\boldsymbol{\varphi}(x)$ 及 $\boldsymbol{u}_{mt}(x,0)\to\boldsymbol{\psi}(x)$ 在 $H_0^M(\Omega)$ 中强收敛，若条件(1.5)，(1.6)式成立，则有估计：

$$\left|\boldsymbol{u}_m\right|_{L^2(\Omega)}^2+\left|\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_m\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left\|\nabla^M\boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2\le C_1\ \left(C_1\text{为与}\boldsymbol{u}_m\text{无关的正常数}\right).$$

引理 3.2：设 $\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega),\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$, 选取 a_{mij},b_{mij}，使当 $m\to+\infty$ 时，$\boldsymbol{u}_m(x,0)\to\boldsymbol{\varphi}(x)$ 及 $\boldsymbol{u}_{mt}(x,0)\to\boldsymbol{\psi}(x)$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，若条件(1.5)，(1.6)式成立，则有估计：

$$\left|\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left\|\nabla^M\boldsymbol{u}_{mtt}\right\|_{L^2(\Omega)}^2\le C_2\ \left(C_2\text{为与}\boldsymbol{u}_m\text{无关的正常数}\right).$$

引理 3.3：设 $\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega),\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$, 选取 a_{mij},b_{mij} 使当 $m\to+\infty$ 时，$\boldsymbol{u}_m(x,0)\to\boldsymbol{\varphi}(x)$ 及 $\boldsymbol{u}_{mt}(x,0)\to\boldsymbol{\psi}(x)$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，若条件(1.5)，(1.6)式成立，则有估计：

$$\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left|\Delta^M\boldsymbol{u}_m\right|_{L^2(\Omega)}^2+\left|\Delta^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left\|\Delta^M\boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2\le C_3\ \left(C_3\text{为与}\boldsymbol{u}_m\text{无关的正常数}\right).$$

引理 3.4：在引理 3.3 条件下成立不等式

$$\left|\Delta^M\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2\le C_4\ \left(C_4\text{为与}\boldsymbol{u}_m\text{无关的正常数}\right).$$

引理 3.1 的证明：

方程(2.4)两边同乘 $a'_{mik}(t)$

$$\left(u_{mitt},a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mi},a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mit},a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mitt},a'_{mik}(t)w_k\right)$$
$$=\left(f_i(\boldsymbol{u}_{mt}),a'_{mik}(t)w_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt},u_{mit}\right)+\left((-1)^M\Delta^M u_{mi},u_{mit}\right)+\left((-1)^M\Delta^M u_{mit},u_{mit}\right)+\left((-1)^M\Delta^M u_{mitt},u_{mit}\right)=\left(f_i(\boldsymbol{u}_{mt}),u_{mit}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_m,\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mtt},\boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mt}\right),$$

而由条件(1.5)知

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt})-\boldsymbol{f}(\boldsymbol{0}),\boldsymbol{u}_{mt}\right)=\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_{mt})}{\partial\boldsymbol{u}_{mt}}\bigg|_{\theta u_{mt}}\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}\right)\le k_0\left(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}\right)\ \ (0<\theta<1),$$

首先利用格林公式，然后两端从 0 到 t 积分 $(0\le t\le T)$，得

$$\frac{1}{2}(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt})-\frac{1}{2}(\boldsymbol{\psi}_m,\boldsymbol{\psi}_m)+\frac{1}{2}(\nabla^M\boldsymbol{u}_m,\nabla^M\boldsymbol{u}_m)-\frac{1}{2}(\nabla^M\boldsymbol{\varphi}_m,\nabla^M\boldsymbol{\varphi}_m)$$
$$+[\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}]+\frac{1}{2}(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt})-\frac{1}{2}(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m)\le k_0[\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}],$$

这里，$\boldsymbol{\varphi}_m(x)=(\varphi_{m1}(x),\varphi_{m2}(x),\cdots,\varphi_{mN}(x))^{\mathrm{T}}$，$\boldsymbol{\psi}_m(x)=(\psi_{m1}(x),\psi_{m2}(x),\cdots,\psi_{mN}(x))^{\mathrm{T}}$，从而有不等式

$$\frac{1}{2}(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt})+\frac{1}{2}(\nabla^M\boldsymbol{u}_m,\nabla^M\boldsymbol{u}_m)+[\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}]+\frac{1}{2}(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt})$$
$$\le k_0[\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}]+\frac{1}{2}(\boldsymbol{\psi}_m,\boldsymbol{\psi}_m)+\frac{1}{2}(\nabla^M\boldsymbol{\varphi}_m,\nabla^M\boldsymbol{\varphi}_m)+\frac{1}{2}(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m),$$

两边加上 $[\boldsymbol{u}_m,\boldsymbol{u}_{mt}]$,

左边将它估计为 $\frac{1}{2}(\boldsymbol{u}_m,\boldsymbol{u}_m)-\frac{1}{2}(\boldsymbol{\varphi}_m,\boldsymbol{\varphi}_m)$,

右边将它化为 $[\boldsymbol{u}_m,\boldsymbol{u}_{mt}]\le[\boldsymbol{u}_m,\boldsymbol{u}_m]+[\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}]$,

从而有

$$\frac{1}{2}(\boldsymbol{u}_m,\boldsymbol{u}_m)+\frac{1}{2}(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt})+\frac{1}{2}(\nabla^M\boldsymbol{u}_m,\nabla^M\boldsymbol{u}_m)+[\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}]+\frac{1}{2}(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt})$$
$$\le\frac{1}{2}(\boldsymbol{\varphi}_m,\boldsymbol{\varphi}_m)+\frac{1}{2}(\boldsymbol{\psi}_m,\boldsymbol{\psi}_m)+\frac{1}{2}(\nabla^M\boldsymbol{\varphi}_m,\nabla^M\boldsymbol{\varphi}_m)+\frac{1}{2}(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m)+(k_0+1)[\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}]+[\boldsymbol{u}_m,\boldsymbol{u}_m]$$

由于当 $m\to+\infty$ 时，$\boldsymbol{\varphi}_m\to\boldsymbol{\varphi}$ 及 $\boldsymbol{\psi}_m\to\boldsymbol{\psi}$ 在 $H_0^M(\Omega)$ 中强收敛，所以当 $m\to+\infty$ 时，$(\boldsymbol{\varphi}_m,\boldsymbol{\varphi}_m)\to(\boldsymbol{\varphi},\boldsymbol{\varphi})$，$(\boldsymbol{\psi}_m,\boldsymbol{\psi}_m)\to(\boldsymbol{\psi},\boldsymbol{\psi})$，$(\nabla^M\boldsymbol{\varphi}_m,\nabla^M\boldsymbol{\varphi}_m)\to(\nabla^M\boldsymbol{\varphi},\nabla^M\boldsymbol{\varphi})$ 及 $(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m)\to(\nabla^M\boldsymbol{\psi},\nabla^M\boldsymbol{\psi})$。以上不等式右边前四项能用一与 m 无关的正常数来控制。由 Gronwall 不等式得

$$|\boldsymbol{u}_m|^2_{L^2(\Omega)}+|\boldsymbol{u}_{mt}|^2_{L^2(\Omega)}+|\nabla^M\boldsymbol{u}_m|^2_{L^2(\Omega)}+|\nabla^M\boldsymbol{u}_{mt}|^2_{L^2(\Omega)}+\|\nabla^M\boldsymbol{u}_{mt}\|^2_{L^2(\Omega)}\le C_1$$

引理 3.1 证毕!

引理 3.2 的证明：

对方程组(2.4)两边关于 t 求导得

$$(u_{mittt},w_k)+((-1)^M\Delta^M u_{mit},w_k)+((-1)^M\Delta^M u_{mitt},w_k)+((-1)^M\Delta^M u_{mittt},w_k)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i(\boldsymbol{u}_{mt}),w_k\right)$$

两边同乘以 $a''_{mik}(t)$ 得

$$(u_{mittt},a''_{mik}(t)w_k)+((-1)^M\Delta^M u_{mit},a''_{mik}(t)w_k)+((-1)^M\Delta^M u_{mitt},a''_{mik}(t)w_k)$$
$$+((-1)^M\Delta^M u_{mittt},a''_{mik}(t)w_k)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i(\boldsymbol{u}_{mt}),a''_{mik}(t)w_k\right)$$

关于 k 从 1 到 m 作和得

$$(u_{mittt},u_{mitt})+((-1)^M\Delta^M u_{mit},u_{mitt})+((-1)^M\Delta^M u_{mitt},u_{mitt})+((-1)^M\Delta^M u_{mittt},u_{mitt})=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i(\boldsymbol{u}_{mt}),u_{mitt}\right)$$

关于 i 从 1 到 N 作和得

$$(\boldsymbol{u}_{mttt},\boldsymbol{u}_{mtt})+((-1)^M\Delta^M\boldsymbol{u}_{mt},\boldsymbol{u}_{mtt})+((-1)^M\Delta^M\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})+((-1)^M\Delta^M\boldsymbol{u}_{mttt},\boldsymbol{u}_{mtt})=\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mtt}\right)$$

而由条件(1.5)知 $\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mtt}\right)=\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_{mt})}{\partial\boldsymbol{u}_{mt}}\cdot\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)\le k_0(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})$,

从而有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt})+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt})+(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt})\le k_0(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})$$

两边从 0 到 $t(0\le t\le T)$ 积分得

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})-\frac{1}{2}(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0))+\frac{1}{2}(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt})-\frac{1}{2}(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m)\\&+\frac{1}{2}(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt})-\frac{1}{2}(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0))+[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}]\le k_0[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}]\end{aligned}$$

下证 $(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0))+(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0))$ 有界(能用一与 m 无关正常数控制住)。

方程组(2.4)两边同乘以 $a''_{mik}(t)$ 得

$$\begin{aligned}&(u_{mitt},a''_{mik}(t)w_k)+((-1)^M\Delta^M u_{mi},a''_{mik}(t)w_k)+((-1)^M\Delta^M u_{mit},a''_{mik}(t)w_k)+((-1)^M\Delta^M u_{mitt},a''_{mik}(t)w_k)\\&=(f_i(\boldsymbol{u}_{mt}),a''_{mik}(t)w_k),\end{aligned}$$

关于 k 从 1 到 m 作和得

$$(u_{mitt},u_{mitt})+((-1)^M\Delta^M u_{mi},u_{mitt})+((-1)^M\Delta^M u_{mit},u_{mitt})+((-1)^M\Delta^M u_{mitt},u_{mitt})=(f_i(\boldsymbol{u}_{mt}),u_{mitt})$$

关于 i 从 1 到 N 作和得

$$(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})+((-1)^M\Delta^M\boldsymbol{u}_m,\boldsymbol{u}_{mtt})+((-1)^M\Delta^M\boldsymbol{u}_{mt},\boldsymbol{u}_{mtt})+((-1)^M\Delta^M\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt})=(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mtt})$$

令 $t=0$ 得

$$\begin{aligned}&(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0))+((-1)^M\Delta^M\boldsymbol{u}_m(x,0),\boldsymbol{u}_{mtt}(x,0))+((-1)^M\Delta^M\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mtt}(x,0))\\&+((-1)^M\Delta^M\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0))=(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{u}_{mtt}(x,0))\end{aligned}\tag{**}$$

$$(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{u}_{mtt}(x,0))\le\frac{1}{2\varepsilon_1}(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m))+\frac{\varepsilon_1}{2}(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0))$$

(其中 $\boldsymbol{\psi}_m=(\psi_{m1}(x),\cdots,\psi_{mN}(x))^{\mathrm{T}}$),

下面证明 $(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m))$ 有界。

由于

$$(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m))\le\left(a_1+b_1|\boldsymbol{\psi}_m|^{\frac{p}{2}},a_1+b_1|\boldsymbol{\psi}_m|^{\frac{p}{2}}\right)=a_1^2|\Omega|+2a_1b_1\int_\Omega|\boldsymbol{\psi}_m|^{\frac{p}{2}}\mathrm{d}x+b_1^2\int_\Omega|\boldsymbol{\psi}_m|^p\mathrm{d}x.$$

当 $m\to+\infty$ 时，$\boldsymbol{\psi}_m\to\boldsymbol{\psi}$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，由 Sobolev 嵌入定理[5]可知 $|\boldsymbol{\psi}_m|_{L^p(\Omega)}\le\text{const}$，其中当 $2M<n$ 时，$2\le p<\frac{2n}{n-2M}$；当 $2M\ge n$ 时，$2\le p<+\infty$。

所以 $\int_\Omega|\boldsymbol{\psi}_m|^p\mathrm{d}x=|\boldsymbol{\psi}_m|^p_{L^p(\Omega)}\le\text{const}$，

从而有

$$\int_\Omega|\boldsymbol{\psi}_m|^{\frac{p}{2}}\mathrm{d}x\le\left(\int_\Omega|\boldsymbol{\psi}_m|^{\frac{p}{2}\times2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega1^2\mathrm{d}x\right)^{\frac{1}{2}}=|\Omega|^{\frac{1}{2}}|\boldsymbol{\psi}_m|^{\frac{p}{2}}_{L^p(\Omega)}\le\text{const},$$

因此，$\left(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m)\right)$ 有界。

将 $(**)$ 中的项 $\left((-1)^M\Delta^M\boldsymbol{u}_m(x,0),\boldsymbol{u}_{mtt}(x,0)\right)$ 移至等式右边得

$$-\left((-1)^M\Delta^M\boldsymbol{u}_m(x,0),\boldsymbol{u}_{mtt}(x,0)\right)=\left((-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0),\boldsymbol{u}_{mtt}(x,0)\right),$$

应用带 ε 的柯西不等式得

$$\left((-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\le\frac{1}{2\varepsilon_2}\left((-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0)\right)+\frac{\varepsilon_2}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)$$

故 $(**)$ 可变为

$$\begin{aligned}&\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\le\left(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{u}_{mtt}(x,0)\right)+\frac{1}{2\varepsilon_2}\left((-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0)\right)+\frac{\varepsilon_2}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right),\end{aligned}$$

再将项 $\left((-1)^M\Delta^M\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)$ 移至等式右边得

$$-\left((-1)^M\Delta^M\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)=\left((-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right),$$

应用带 ε 的柯西不等式得

$$\left((-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\le\frac{1}{2\varepsilon_3}\left((-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0)\right)+\frac{\varepsilon_3}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)$$

故 $(**)$ 可变为

$$\begin{aligned}&\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m)\right)+\frac{\varepsilon_1}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\quad+\frac{1}{2\varepsilon_2}\left((-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0)\right)+\frac{\varepsilon_2}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\quad+\frac{1}{2\varepsilon_3}\left((-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0)\right)+\frac{\varepsilon_3}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right),\end{aligned}$$

因为 $\Delta^M\boldsymbol{u}_m(x,0)=\Delta^M\boldsymbol{\varphi}_m$， $\Delta^M\boldsymbol{u}_{mt}(x,0)=\Delta^M\boldsymbol{\psi}_m$ 所以在上式中

$$\left((-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_m(x,0)\right)=\left((-1)^{M+1}\Delta^M\boldsymbol{\varphi}_m,(-1)^{M+1}\Delta^M\boldsymbol{\varphi}_m\right),$$

$$\left((-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0),(-1)^{M+1}\Delta^M\boldsymbol{u}_{mt}(x,0)\right)=\left((-1)^{M+1}\Delta^M\boldsymbol{\psi}_m,(-1)^{M+1}\Delta^M\boldsymbol{\psi}_m\right),$$

从而有

$$\begin{aligned}&\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m)\right)+\frac{\varepsilon_1}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\quad+\frac{1}{2\varepsilon_2}\left((-1)^{M+1}\Delta^M\boldsymbol{\varphi}_m(x,0),(-1)^{M+1}\Delta^M\boldsymbol{\varphi}_m(x,0)\right)+\frac{\varepsilon_2}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\quad+\frac{1}{2\varepsilon_3}\left((-1)^{M+1}\Delta^M\boldsymbol{\psi}_m(x,0),(-1)^{M+1}\Delta^M\boldsymbol{\psi}_m(x,0)\right)+\frac{\varepsilon_3}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right),\end{aligned}$$

移项整理得

$$\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}\right)\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)$$
$$\le\frac{1}{2\varepsilon_2}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)+\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m)\right),$$

取 $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 充分小，使得 $\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}\right)\ge\frac{1}{2}$。

由于当 $m\to+\infty$ 时，$\boldsymbol{\psi}_m\to\boldsymbol{\psi}$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，故当 $m\to+\infty$ 时，$\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)\to\left(\Delta^M\boldsymbol{\psi},\Delta^M\boldsymbol{\psi}\right)$，故 $\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)$ 能用一与 m 无关的正常数控制住，同样道理 $\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)$ 也能用一与 m 无关的正常数控制住，所以 $\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)$ 有界。

又由已证不等式

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right)+\left[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right]$$
$$\le\frac{1}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)+k_0\left[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right],$$

因为 $\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)$ 有界，$\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)$ 又能用一与 m 无关的正常数控制住，所以

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right)+\left[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right]\le k_0\left[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right]+C,$$

其中 C 为与 m 无关的正常数，再根据 Gronwall 不等式得

$$\left|\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left\|\nabla^M\boldsymbol{u}_{mtt}\right\|_{L^2(\Omega)}^2\le C_2.$$

引理 3.2 证毕！

引理 3.3 的证明：

方程(2.4)两边同乘 $\lambda_k a'_{mik}(t)$ 得

$$\left(u_{mitt},\lambda_k a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mi},\lambda_k a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mit},\lambda_k a'_{mik}(t)w_k\right)$$
$$+\left((-1)^M\Delta^M u_{mitt},\lambda_k a'_{mik}(t)w_k\right)=\left(f_i(\boldsymbol{u}_{mt}),\lambda_k a'_{mik}(t)w_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt},(-1)^M\Delta^M u_{mit}\right)+\left((-1)^M\Delta^M u_{mi},(-1)^M\Delta^M u_{mit}\right)+\left((-1)^M\Delta^M u_{mit},(-1)^M\Delta^M u_{mit}\right)$$
$$+\left((-1)^M\Delta^M u_{mitt},(-1)^M\Delta^M u_{mit}\right)=\left(f_i(\boldsymbol{u}_{mt}),(-1)^M\Delta^M u_{mit}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_m,(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)$$
$$+\left((-1)^M\Delta^M\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M\boldsymbol{u}_{mt}\right),$$

对于右边项经计算得

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right)+\frac{\varepsilon_1}{2}\left((-1)^M\Delta^M\boldsymbol{u}_{mt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right),$$

取 ε_1 充分小，使得 $1-\frac{\varepsilon_1}{2}\ge\frac{1}{2}$ 故成立如下不等式

$$\begin{aligned}&\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{m},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left((-1)^M\Delta^M\boldsymbol{u}_{mt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)\\&+\left((-1)^M\Delta^M\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right),\end{aligned}$$

由格林公式得

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta^M\boldsymbol{u}_{m},\Delta^M\boldsymbol{u}_{m}\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\\&+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right),\end{aligned}$$

两边从 0 到 $t\,(0\le t\le T)$ 积分并移项得

$$\begin{aligned}&\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{m},\Delta^M\boldsymbol{u}_{m}\right)+\frac{1}{2}\left[\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right]+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\\&\le\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_{m},\nabla^M\boldsymbol{\psi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_{m},\Delta^M\boldsymbol{\varphi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_{m},\Delta^M\boldsymbol{\psi}_{m}\right)+\frac{1}{2\varepsilon_1}\left[\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right],\end{aligned}$$

由引理 3.1 知 $\left|\boldsymbol{u}_{mt}\right|_{H^M(\Omega)}^2\le\text{const}\ (0\le t\le T)$，

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right)\le\left(a_1+b_1\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}},a_1+b_1\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\right)=a_1\left|\Omega\right|+2a_1b_1\int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\mathrm{d}x+b_1^2\int_\Omega\left|\boldsymbol{u}_{mt}\right|^p\mathrm{d}x,$$

由 Sobolev 嵌入定理[5]得

$$\int_\Omega\left|\boldsymbol{u}_{mt}\right|^p\mathrm{d}x=\left|\boldsymbol{u}_{mt}\right|_{L^p(\Omega)}^p\le\text{const},\ \int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\mathrm{d}x\le\left(\int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}\times2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega1^2\mathrm{d}x\right)^{\frac{1}{2}}=\left|\Omega\right|^{\frac{1}{2}}\left|\boldsymbol{u}_{mt}\right|_{L^p(\Omega)}^{\frac{p}{2}}\le\text{const}$$

其中当 $2M<n$ 时，$2\le p<\frac{2n}{n-2M}$；当 $2M\ge n$ 时，$2\le p<+\infty$。

所以 $\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right)\le\text{const}$。在下面的不等式中，

$$\begin{aligned}&\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{m},\Delta^M\boldsymbol{u}_{m}\right)+\frac{1}{2}\left[\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right]+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\\&\le\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_{m},\nabla^M\boldsymbol{\psi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_{m},\Delta^M\boldsymbol{\varphi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_{m},\Delta^M\boldsymbol{\psi}_{m}\right)+\frac{1}{2\varepsilon_1}\left[\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right],\end{aligned}$$

由于当 $m\to+\infty$ 时，$\boldsymbol{\varphi}_m\to\boldsymbol{\varphi}$，$\boldsymbol{\psi}_m\to\boldsymbol{\psi}$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，所以当 $m\to+\infty$ 时，

$$\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_{m},\nabla^M\boldsymbol{\psi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_{m},\Delta^M\boldsymbol{\varphi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_{m},\Delta^M\boldsymbol{\psi}_{m}\right)$$

收敛于 $\frac{1}{2}\left(\nabla^M\boldsymbol{\psi},\nabla^M\boldsymbol{\psi}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi},\Delta^M\boldsymbol{\varphi}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\psi},\Delta^M\boldsymbol{\psi}\right)$，

$\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_{m},\nabla^M\boldsymbol{\psi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_{m},\Delta^M\boldsymbol{\varphi}_{m}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_{m},\Delta^M\boldsymbol{\psi}_{m}\right)$ 能用一与 m 无关的正常数控制住，因此由以上不等式得

$$\frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mt},\nabla^M \boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_{m},\Delta^M \boldsymbol{u}_{m}\right)+\frac{1}{2}\left[\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right]+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right)\le \text{const},$$

即

$$\left|\nabla^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left|\Delta^M \boldsymbol{u}_{m}\right|_{L^2(\Omega)}^2+\left|\Delta^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left\|\Delta^M \boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2\le C_3\ \left(C_3\text{为与}\boldsymbol{u}_m\text{无关的正常数}\right).$$

引理 3.3 证毕！

引理 3.4 的证明：

方程(2.4)两边同乘以 $\lambda_k a''_{mik}(t)$，得

$$\begin{aligned}&\left(u_{mitt},\lambda_k a''_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mi},\lambda_k a''_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mit},\lambda_k a''_{mik}(t)w_k\right)\\&+\left((-1)^M\Delta^M u_{mitt},\lambda_k a''_{mik}(t)w_k\right)=\left(f_i(\boldsymbol{u}_{mt}),\lambda_k a''_{mik}(t)w_k\right),\end{aligned}$$

关于 k 从 1 到 m 作和得

$$\begin{aligned}&\left(u_{mitt},(-1)^M\Delta^M u_{mitt}\right)+\left((-1)^M\Delta^M u_{mi},(-1)^M\Delta^M u_{mitt}\right)+\left((-1)^M\Delta^M u_{mit},(-1)^M\Delta^M u_{mitt}\right)\\&+\left((-1)^M\Delta^M u_{mitt},(-1)^M\Delta^M u_{mitt}\right)=\left(f_i(\boldsymbol{u}_{mt}),(-1)^M\Delta^M u_{mitt}\right),\end{aligned}$$

关于 i 从 1 到 N 作和得

$$\begin{aligned}&\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)+\left((-1)^M\Delta^M \boldsymbol{u}_{m},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)+\left((-1)^M\Delta^M \boldsymbol{u}_{mt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)\\&+\left((-1)^M\Delta^M \boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right),\end{aligned}$$

移项得如下不等式

$$\begin{aligned}&\left((-1)^M\Delta^M \boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)=-\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)-\left((-1)^M\Delta^M \boldsymbol{u}_{m},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)\\&-\left((-1)^M\Delta^M \boldsymbol{u}_{mt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)+\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right),\end{aligned}$$

而

$$\begin{aligned}&-\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)\le\frac{\varepsilon_1}{2}\left((-1)^M\Delta^M \boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)\\&=\frac{\varepsilon_1}{2}\left(\Delta^M \boldsymbol{u}_{mtt},\Delta^M \boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right),\end{aligned}$$

$$\begin{aligned}&-\left((-1)^M\Delta^M \boldsymbol{u}_{m},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)\\&\le\frac{\varepsilon_2}{2}\left((-1)^M\Delta^M \boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_2}\left((-1)^M\Delta^M \boldsymbol{u}_{m},(-1)^M\Delta^M \boldsymbol{u}_{m}\right)\\&=\frac{\varepsilon_2}{2}\left(\Delta^M \boldsymbol{u}_{mtt},\Delta^M \boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_2}\left(\Delta^M \boldsymbol{u}_{m},\Delta^M \boldsymbol{u}_{m}\right),\end{aligned}$$

$$\begin{aligned}&-\left((-1)^M\Delta^M \boldsymbol{u}_{mt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)\\&\le\frac{\varepsilon_3}{2}\left((-1)^M\Delta^M \boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_3}\left((-1)^M\Delta^M \boldsymbol{u}_{mt},(-1)^M\Delta^M \boldsymbol{u}_{mt}\right)\\&=\frac{\varepsilon_3}{2}\left(\Delta^M \boldsymbol{u}_{mtt},\Delta^M \boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right),\end{aligned}$$

$$\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),(-1)^M\Delta^M\boldsymbol{u}_{mtt}\right)\le\frac{\varepsilon_4}{2}\left((-1)^M\Delta^M\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_4}\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right),$$

所以

$$\begin{aligned}\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)&=\left((-1)^M\Delta^M\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mtt}\right)\\&\le\frac{\varepsilon_1}{2}\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{\varepsilon_2}{2}\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)\\&\quad+\frac{1}{2\varepsilon_2}\left(\Delta^M\boldsymbol{u}_{m},\Delta^M\boldsymbol{u}_{m}\right)+\frac{\varepsilon_3}{2}\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\\&\quad+\frac{\varepsilon_4}{2}\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_4}\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right),\end{aligned}$$

经计算得

$$\begin{aligned}&\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}-\frac{\varepsilon_4}{2}\right)\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)\\&\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_2}\left(\Delta^M\boldsymbol{u}_{m},\Delta^M\boldsymbol{u}_{m}\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2\varepsilon_4}\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right)\end{aligned}$$

由引理 3.2 和引理 3.3 的结论知

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)\le\text{const},\ \left(\Delta^M\boldsymbol{u}_{m},\Delta^M\boldsymbol{u}_{m}\right)\le\text{const},\ \left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\le\text{const}$$

类似于引理 3.2 的证明过程可证 $\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right)\le\text{const},$

取 $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ 充分小，使得 $\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}-\frac{\varepsilon_4}{2}\right)\ge\frac{1}{2}$，所以

$$\left(\Delta^M\boldsymbol{u}_{mtt},\Delta^M\boldsymbol{u}_{mtt}\right)\le\text{const},\text{即}\left|\Delta^M\boldsymbol{u}_{mtt}\right|^2_{L^2(\Omega)}\le C_4\quad\left(C_4\text{为与}\boldsymbol{u}_m\text{无关的正常数}\right).$$

引理 3.4 证毕！

下面在引理 3.1~3.4 的基础上给出定理 1 的证明。

定理 1 的证明：将(2.1)~(2.3)代入(2.4)及其初值条件得

$$\begin{cases}\sum_{j=1}^{m}a''_{mij}(t)\left(w_j,w_k\right)+\sum_{j=1}^{m}\left((-1)^M\Delta^Mw_j,w_k\right)a_{mij}(t)+\sum_{j=1}^{m}\left((-1)^M\Delta^Mw_j,w_k\right)a'_{mij}(t)\\\qquad+\sum_{j=1}^{m}\left((-1)^M\Delta^Mw_j,w_k\right)a''_{mij}(t)=\left(f_i\left(u_{mt}\right),w_k\right)\\a_{mij}(0)=a_{mij}\\a'_{mij}(0)=b_{mij}\\k=1,2,\cdots,m;\ i=1,2,\cdots,N.\end{cases}$$

由引理 3.1~引理 3.4 的证明过程知 $\left|\left(f_i\left(\boldsymbol{u}_{mt}\right),w_k\right)\right|$ 有界，因此由常微分方程理论知上述常微分方程组在 $[0,T]$ 上有整体解，从而常微分方程组 $(*)$ 在 $[0,T]$ 上有整体解 $u_{mi}(x,t)$，且有下列估计式成立：

$$\left|\boldsymbol{u}_m\right|^2_{L^2(\Omega)}\le\text{const},\quad\left|\boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)}\le\text{const},\quad\left|\boldsymbol{u}_{mtt}\right|^2_{L^2(\Omega)}\le\text{const},\ \left\|\nabla^M\boldsymbol{u}_{mtt}\right\|^2_{L^2(\Omega)}\le\text{const},$$

$$\left|\nabla^M\boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)}\le\text{const},\ \left|\Delta^M\boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)}\le\text{const},\ \left|\Delta^M\boldsymbol{u}_{m}\right|^2_{L^2(\Omega)}\le\text{const}.$$

由列紧性原理知存在 $\{u_{mi}\}$ 的一个子序列，不妨设为 $\{u_{vi}\}$，使得当 $v\to\infty$ 时，u_{vi} 弱收敛到 u_i 于 $L^\infty\left([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega)\right)$，$u_{vit}$ 弱收敛到 u_{it} 于

$L^\infty\left([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega)\right)\subset L^\infty\left([0,T],L^2(\Omega)\right)$，$u_{vitt}$ 弱收敛到 u_{itt} 于

$L^2\left([0,T],H_0^M(\Omega)\right)\cap L^\infty\left([0,T],L^2(\Omega)\right)$，另一方面，由内插不等式知 $|\nabla\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 可被 $|\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 与 $|\nabla^M\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 控制住，而 $|\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 与 $|\nabla^M\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 是有界的，所以 $|\nabla\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 有界。故由引理 3.1~引理 3.4 知 $\boldsymbol{u}_{vt},\boldsymbol{u}_{vtt},\nabla\boldsymbol{u}_{vt}$ 于 $L^\infty\left(0,T;L^2(\Omega)\right)\subset L^2\left(0,T;L^2(\Omega)\right)=L^2(Q_T)\left(Q_T=\Omega\times(0,T)\right)$ 中有界，因此 $\boldsymbol{u}_{vt}$ 在 $H^1(Q_T)$ 中有界，由 $H^1(Q_T)$ 紧致嵌入到 $L^2(Q_T)$ 知从 $\boldsymbol{u}_{vt}$ 可选取一子序列(仍记为 $\boldsymbol{u}_{vt}$)在 $L^2(Q_T)$ 中强收敛且几乎处处收敛到 $\boldsymbol{u}_t$。又由引理 3.3 的证明过程知 $\left(f(\boldsymbol{u}_{vt}),f(\boldsymbol{u}_{vt})\right)=\left|f(\boldsymbol{u}_{vt})\right|_{L^2(\Omega)}^2\le\text{const}$，由[6]中引理 1.3 知 $f(\boldsymbol{u}_{vt})\to f(\boldsymbol{u}_t)$ 在 $L^2(Q_T)$ 中弱收敛。在(2.4)中令 $m=v$ 将

$$\left(u_{mitt},w_k\right)+\left((-1)^M\Delta^M u_{mi},w_k\right)+\left((-1)^M\Delta^M u_{mit},w_k\right)+\left((-1)^M\Delta^M u_{mitt},w_k\right)=\left(f_i(\boldsymbol{u}_{mt}),w_k\right),$$

改为

$$\left(u_{vitt},w_k\right)+\left((-1)^M\Delta^M u_{vi},w_k\right)+\left((-1)^M\Delta^M u_{vit},w_k\right)+\left((-1)^M\Delta^M u_{vitt},w_k\right)=\left(f_i(\boldsymbol{u}_{vt}),w_k\right),$$

两边同乘 $d_{ki}(t)$，其中 $d_{ki}(t)\in C[0,T],k=1,2,\cdots;i=1,2,\cdots,N$，得

$$\begin{aligned}&\left(u_{vitt},d_{ki}(t)w_k\right)+\left((-1)^M\Delta^M u_{vi},d_{ki}(t)w_k\right)+\left((-1)^M\Delta^M u_{vit},d_{ki}(t)w_k\right)+\left((-1)^M\Delta^M u_{vitt},d_{ki}(t)w_k\right)\\&=\left(f_i(\boldsymbol{u}_{vt}),d_{ki}(t)w_k\right),\end{aligned}$$

对 $k=1,2,\cdots,v'\ (v'\le v)$ 求和得

$$\begin{aligned}&\left(u_{vitt},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)+\left((-1)^M\Delta^M u_{vi},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)+\left((-1)^M\Delta^M u_{vit},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\\&+\left((-1)^M\Delta^M u_{vitt},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)=\left(f_i(\boldsymbol{u}_{vt}),\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right),\end{aligned}$$

对 t 从 0 到 T 积分得

$$\begin{aligned}&\int_0^T\left(u_{vitt},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t+\int_0^T\left((-1)^M\Delta^M u_{vi},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t+\int_0^T\left((-1)^M\Delta^M u_{vit},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t\\&+\int_0^T\left((-1)^M\Delta^M u_{vitt},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t=\int_0^T\left(f_i(\boldsymbol{u}_{vt}),\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t,\end{aligned}$$

对应当 $v\to+\infty$ 时，$\boldsymbol{f}(\boldsymbol{u}_{vt})\to\boldsymbol{f}(\boldsymbol{u}_t)$ 在 $L^2(Q_T)$ 中弱收敛，有当 $v\to+\infty$ 时，$f_i(\boldsymbol{u}_{vt})\to f_i(\boldsymbol{u}_t)$ 在 $L^2(Q_T)$ 中弱收敛 $(i=1,2,\cdots,N)$，令 $v\to+\infty$ 得

$$\begin{aligned}&\int_0^T\left(u_{itt},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t+\int_0^T\left((-1)^M\Delta^M u_i,\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t+\int_0^T\left((-1)^M\Delta^M u_{it},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t\\&+\int_0^T\left((-1)^M\Delta^M u_{itt},\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t=\int_0^T\left(f_i(\boldsymbol{u}_t),\sum_{k=1}^{v'}d_{ki}(t)w_k(x)\right)\mathrm{d}t,\end{aligned}$$

因为 $\{w_k(x)\}_{k=1}^{+\infty}$ 构成 $L^2(\Omega)$ 的一组标准正交基，

$$d_{ki}(t)\in C[0,T],k=1,2,\cdots;i=1,2,\cdots,N$$

$\left\{\sum_{k=1}^{v'} d_{ki}(t) w_k(x) \middle| v'=1,2,\cdots; i=1,2,\cdots,N\right\}$ 在空间 $C([0,T];L^2(\Omega))$ 中稠密，所以对于任意的 $d_i(x,t)\in C([0,T];L^2(\Omega))$ $(i=1,2,\cdots,N)$ 成立

$$\int_0^T (u_{itt}, d_i(x,t))\mathrm{d}t + \int_0^T ((-1)^M \Delta^M u_i, d_i(x,t))\mathrm{d}t + \int_0^T ((-1)^M \Delta^M u_{it}, d_i(x,t))\mathrm{d}t$$
$$+\int_0^T ((-1)^M \Delta^M u_{itt}, d_i(x,t))\mathrm{d}t = \int_0^T (f_i(\boldsymbol{u}_t), d_i(x,t))\mathrm{d}t,$$

上式关于 $i=1,2,\cdots,N$ 求和得对任意

$$\boldsymbol{d}(x,t)=(d_1(x,t),d_2(x,t),\cdots,d_N(x,t))^{\mathrm{T}}\in C([0,T];L^2(\Omega)) \text{ 成立}$$

$$\int_0^T (\boldsymbol{u}_{tt}, \boldsymbol{d}(x,t))\mathrm{d}t + \int_0^T ((-1)^M \Delta^M \boldsymbol{u}, \boldsymbol{d}(x,t))\mathrm{d}t + \int_0^T ((-1)^M \Delta^M \boldsymbol{u}_t, \boldsymbol{d}(x,t))\mathrm{d}t$$
$$+\int_0^T ((-1)^M \Delta^M \boldsymbol{u}_{tt}, \boldsymbol{d}(x,t))\mathrm{d}t = \int_0^T (\boldsymbol{f}(\boldsymbol{u}_t), \boldsymbol{d}(x,t))\mathrm{d}t.$$

下面验证初始条件 $\boldsymbol{u}(x,0)=\boldsymbol{\varphi}(x)$。因为当 $v\to\infty$ 时，u_{vi} 弱收敛到 u_i 于 $L^\infty([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega))$，$u_{vit}$ 弱收敛到 u_{it} 于

$$L^\infty([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega))\subset L^\infty([0,T],L^2(\Omega)),$$

因此 $u_{vi}(x,t), u_i(x,t)\in C([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega))$，

所以，当 $v\to+\infty$ 时 $u_{vi}(x,0)$ 弱收敛到 $u_i(x,0)$ 于 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中，而当 $v\to+\infty$ 时，$u_{vi}(x,0)\to\varphi_i(x)$ $(i=1,2,\cdots,N)$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，故

$$u_i(x,0)=\varphi_i(x)\ (1,2,\cdots,N)\text{，即 }\boldsymbol{u}(x,0)=\boldsymbol{\varphi}(x)\text{。}$$

再证初始条件 $\boldsymbol{u}_t(x,0)=\boldsymbol{\psi}(x)$。因为当 $v\to\infty$ 时 u_{vit} 弱收敛到 u_{it} 于 $L^\infty([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega))\subset L^\infty([0,T],L^2(\Omega))$，$u_{vitt}$ 弱收敛到 u_{itt} 于 $L^2([0,T],H_0^M(\Omega))\cap L^\infty([0,T],L^2(\Omega))$，因此 $u_{vit}(x,t),u_{it}(x,t)\in C([0,T],L^2(\Omega))$，所以，$u_{vit}(x,0)$ 弱收敛到 $u_{it}(x,0)$ $(i=1,2,\cdots,N)$ 于 $L^2(\Omega)$ 中，而当 $v\to+\infty$ 时，$u_{vit}(x,0)\to\psi_i(x)$ $(i=1,2,\cdots,N)$ 在 $L^2(\Omega)$ 中强收敛，故 $u_{it}(x,0)=\psi_i(x)$ $(i=1,2,\cdots,N)$，即 $\boldsymbol{u}_t(x,0)=\boldsymbol{\psi}(x)$。

综上所述由定义 $\boldsymbol{u}=\boldsymbol{u}(x,t)$ 为问题(1.1)~(1.3)的整体强解。

下面给出定理 2 的证明，定理 2 是关于问题(1.1)~(1.3)强解唯一性的讨论。

定理 2 的证明：设 $\boldsymbol{u}^1,\boldsymbol{u}^2$ 为非线性问题(1.1)~(1.3)的两个整体强解，

令 $\boldsymbol{w}=\boldsymbol{u}^1-\boldsymbol{u}^2$，则 $\boldsymbol{w}$ 满足

$$\begin{cases}\boldsymbol{w}_{tt}+(-1)^M\Delta^M\boldsymbol{w}+(-1)^M\Delta^M\boldsymbol{w}_t+(-1)^M\Delta^M\boldsymbol{w}_{tt}=\boldsymbol{f}(\boldsymbol{u}_t^1)-\boldsymbol{f}(\boldsymbol{u}_t^2),\\ \boldsymbol{w}(x,0)=0,\boldsymbol{w}_t(x,0)=0,\\ D^\gamma\boldsymbol{w}(x,t)\big|_{\partial\Omega\times[0,T]}=0, \qquad 0\le|\gamma|\le M-1.\end{cases}$$

两边用 $\boldsymbol{w}_t$ 做内积，得

$$(\boldsymbol{w}_{tt},\boldsymbol{w}_t)+((-1)^M\Delta^M\boldsymbol{w},\boldsymbol{w}_t)+((-1)^M\Delta^M\boldsymbol{w}_t,\boldsymbol{w}_t)+((-1)^M\Delta^M\boldsymbol{w}_{tt},\boldsymbol{w}_t)$$
$$=\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_t)}{\partial\boldsymbol{u}_t}\bigg|_{\boldsymbol{u}_t^2+\theta\boldsymbol{w}_t}\boldsymbol{w}_t,\boldsymbol{w}_t\right)\qquad(0<\theta<1),$$

由条件(1.5)知 $\left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_t)}{\partial \boldsymbol{u}_t}\Big|_{u_t^2+\theta w_t} \boldsymbol{w}_t, \boldsymbol{w}_t\right) \le k_0(\boldsymbol{w}_t, \boldsymbol{w}_t)\ (0<\theta<1)$，

因此

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}_t,\boldsymbol{w}_t)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w})+(\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t)\le k_0(\boldsymbol{w}_t,\boldsymbol{w}_t),$$

两边从 0 到 $t(0\le t\le T)$ 积分得

$$\frac{1}{2}(\boldsymbol{w}_t,\boldsymbol{w}_t)+\frac{1}{2}(\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w})+[\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t]+\frac{1}{2}(\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t)\le k_0[\boldsymbol{w}_t,\boldsymbol{w}_t],$$

两边同时加上 $[\boldsymbol{w},\boldsymbol{w}_t]+[\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w}_t]$，左边将它化为 $\frac{1}{2}(\boldsymbol{w},\boldsymbol{w})+\frac{1}{2}(\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w})$，右边将它估计为

$$[\boldsymbol{w},\boldsymbol{w}_t]+[\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w}_t]\le[\boldsymbol{w},\boldsymbol{w}]+[\boldsymbol{w}_t,\boldsymbol{w}_t]+\frac{1}{2\varepsilon}[\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w}]+\frac{\varepsilon}{2}[\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t],$$

因此有

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{w},\boldsymbol{w})+\frac{1}{2}(\boldsymbol{w}_t,\boldsymbol{w}_t)+(\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w})+[\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t]+\frac{1}{2}(\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t)\\&\le[\boldsymbol{w},\boldsymbol{w}]+(k_0+1)[\boldsymbol{w}_t,\boldsymbol{w}_t]+\frac{\varepsilon}{2}[\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t]+\frac{1}{2\varepsilon}[\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w}],\end{aligned}$$

取 ε 充分小使得 $1-\frac{\varepsilon}{2}\ge\frac{1}{2}$ 整理得

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{w},\boldsymbol{w})+\frac{1}{2}(\boldsymbol{w}_t,\boldsymbol{w}_t)+(\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w})+\frac{1}{2}[\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t]+\frac{1}{2}(\nabla^M \boldsymbol{w}_t,\nabla^M \boldsymbol{w}_t)\\&\le[\boldsymbol{w},\boldsymbol{w}]+(k_0+1)[\boldsymbol{w}_t,\boldsymbol{w}_t]+\frac{1}{2\varepsilon}[\nabla^M \boldsymbol{w},\nabla^M \boldsymbol{w}],\end{aligned}$$

由 Gronwall 不等式得 $|\boldsymbol{w}|^2_{L^2(\Omega)}+|\boldsymbol{w}_t|^2_{L^2(\Omega)}+|\nabla^M \boldsymbol{w}|^2_{L^2(\Omega)}=0.$

所以 $\boldsymbol{w}\equiv\boldsymbol{0}\ \ a.e\ \ \boldsymbol{u}^1=\boldsymbol{u}^2$。

唯一性证毕！

4. 结论

本文主要研究了在粘弹性力学中具有实际背景的一类高阶 n 维非线性波动方程组。要求方程的非线性项 $\boldsymbol{f}$ 一次连续可微且非线性项 $\boldsymbol{f}$ 满足条件：$\boldsymbol{f}$ 的 Jacobi 矩阵半有界。首先选取高阶调和算子的特征函数系作为一组基，用 Galerkin 方法构造初边值问题的近似解，用 4 个引理对近似解作出一系列的先验估计，在先验估计的基础上，由列紧性原理证明了初边值问题整体强解的存在性，最后证明了初边值问题整体强解的唯一性。其中，选取高阶调和算子的特征函数系作为一组基，该特征函数系分别构成 $L^2(\Omega)$，$H_0^M(\Omega), H^{2M}\cap H_0^M(\Omega)$ 的正交基底，这样对近似解作先验估计时会更方便。

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