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The Stability of a Multi-Quadratic Functional Equation on a Restricted Domain

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Abstract

In this paper, we obtain the stability of the multi-quadratic functional equation on a restricted domain.

Keywords

Hyers-Ulam Stability, Functional Equation, Multi-Quadratic Functional Equation

多元二次函数方程在限制定义域上的稳定性

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摘　要

本文证明了多元二次函数方程在限制定义域上的稳定性。

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关键词

Hyers-Ulam稳定性，函数方程，多元二次函数方程

1. 引言

关于函数方程的稳定性问题，早在 1940 年 S. M. Ulam [1]提出了群同态的稳定性。次年，D.H. Hyers [2]把群 G_1 和 G_2 换做 Banach 空间，并给出近似可加映射的稳定性。在证明这一问题的过程中 Hyers 使用了“直接法”，这一方法是研究各类函数稳定性的有力工具。在 Ulam-Hyers-Rassias 稳定性理论的基础上越来越多的数学家对稳定性理论产生兴趣，从目前研究现状来看，限制定义域上函数方程稳定性问题对于稳定性研究具有重大意义。近几年国内外的许多数学家专注于研究限制定义域上函数方程稳定性理论，而在这方面贡献较为突出的是 F. Skof 和 Jung 等。1983 年 F. Skof 解决 Ulam 可加函数在限制域上稳定性问题。之后又有许多数学家给出了各类函数在限制定义域上稳定性的相关结论：Z. Kominek [3]证明了 Jensen 函数方程在限制定义域上的稳定性；S.M. Jung [4]证明了限制定义域上 Jensen 函数方程稳定性并且应用这一结论研究可加函数的近似性质。John Michael Rassias [5]在 Jung 关于二次函数稳定性证明的基础上，给出了二次函数在限制定义域上的稳定性。Hyers, Isac 和 Rassias [6]给出可加 Cauchy 方程的 Hyers-Ulam-Rassis 稳定性，并应用它去研究渐进可导性。Dorota Wolna [7]证明了多项式函数在限制定义域上的稳定性。John Michael Rassias 和 Matina John Rassias [8]证明了 Jensen 和 Jensen 型函数在限制定义域上的稳定问题，并且给出了 Jensen 和 Jensen 型函数的近似性问题，在证明中用到的方法与文献[5]是一致的。Jae-Young Chung, Dohan Kim 和 John Michael Rassias [9]给出了群上 Jensen 型函数在限制定义域上的稳定性。Yang-Hi Lee [10]在 2013 年证明了限制定义域上二次可加函数方程的稳定性。Won-Gil Park 和 Jae-Hyeong Bae [11]证明了 Bi-二次函数方程稳定性，纪培胜[12]给出了多元二次函数方程等价形式并证明其稳定性。

本文主要证明的是多元二次函数方程在限制定义域上的稳定性。

2. 主要结果及证明

定义 2.1 [12]：函数 $f: X^n \to Y$ 被称作多元二次的或者 n-二次的是指函数 f 关于每一变元都是二次的，即：

$$f\left(x_1,\cdots,x_{i-1},x_i+x_i',x_{i+1},\cdots,x_n\right)+f\left(x_1,\cdots,x_{i-1},x_i-x_i',x_{i+1},\cdots,x_n\right)$$
$$=2f\left(x_1,\cdots,x_n\right)+2f\left(x_1,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n\right)$$

其中 $\forall x_1,\cdots,x_{i-1},x_i,x_i',x_{i+1},\cdots,x_n\in X$ 。

以下均设 X 是赋范线性空间， Y 是赋范 Banach 空间。

引理 2.1 [12]：函数 $f: X^n \to Y$ 对 $\forall x_{11},x_{12}\cdots,x_{n1},x_{n2}\in X$ 满足

$$\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)=2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right) \tag{2.1}$$

当且仅当 f 是多元二次的。

引理 2.2 [12]：设函数 $\phi: X^{2n}\to[0,\infty)$ 满足

$$\Phi\left(x_{11},x_{12},\cdots,x_{n1},x_{n2}\right)=\sum_{k=0}^{\infty}\frac{1}{4^{n(k+1)}}\phi\left(2^k x_{11},2^k x_{12},\cdots,2^k x_{n1},2^k x_{n2}\right)<\infty$$

$\forall x_{11},x_{12},\cdots,x_{n1},x_{n2}\in X$ ， 函数 $f: X^n\to Y$ 对 $\forall x_{11},x_{12},\cdots,x_{n1},x_{n2}\in X$ 满足不等式

$$\left\|\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\right\|\le\phi\left(x_{11},x_{12},\cdots,x_{n1},x_{n2}\right)$$

且对 $\forall\left(x_1,\cdots,x_n\right)\in X^n$，如果 $f\left(x_1,\cdots,x_n\right)=0$，那么 $x_1,\cdots,x_n$ 中至少有一个元素为 0，则存在唯一的多元二次函数 $F:X^n\to Y$ 使得 $\left\|f\left(x_1,\cdots,x_n\right)-F\left(x_1,\cdots,x_n\right)\right\|\le\Phi\left(x_1,x_1,\cdots,x_n,x_n\right)$，$\forall x_1,\cdots,x_n\in X$。

下面来给出并证明方程(2.1)在限制定义域上的稳定性。

定理 2.1 设 $d>0$，$\delta>0$ 是给定的数，函数 $f:X^n\to Y$ 满足

$$\left\|\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\right\|\le\delta \tag{2.2}$$

$\forall x_{11},x_{12},\cdots,x_{n1},x_{n2}\in X$，$\left\|x_{i1}\right\|+\left\|x_{i2}\right\|\ge d$，$i\in\{1,\cdots,n\}$，且对 $\forall\left(x_1,\cdots,x_n\right)\in X^n$，如果 $f\left(x_1,\cdots,x_n\right)=0$，那么 $x_1,\cdots,x_n$ 中至少有一个元素为 0，则存在唯一的多元二次函数 $F:X^n\to Y$ 使得

$$\left\|f\left(x_1,\cdots,x_n\right)-F\left(x_1,\cdots,x_n\right)\right\|\le\frac{17^n\delta}{4^n-1},\quad\forall x_1,\cdots,x_n\in X \tag{2.3}$$

证明： 定理的证明过程分为四部分。

I. 首先证明对 $\forall x_{11},x_{12},\cdots,x_{n1},x_{n2}\in X$

$$\left\|\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\right\|\le 17^{n-1}\delta$$

当 $\left\|x_{11}\right\|+\left\|x_{12}\right\|\ge d$，$\left\|x_{i1}\right\|+\left\|x_{i2}\right\|\ge d$，$i\in\{2,\cdots,n\}$。如果 $x_{11}=x_{12}=0$ 取 $p_1\in X$ 且 $\left\|p_1\right\|=d$，当 $\left\|x_{11}\right\|\ge\left\|x_{12}\right\|$ 时，令 $p_1=\left(1+d/\left\|x_{11}\right\|\right)x_{11}$，当 $\left\|x_{11}\right\|<\left\|x_{12}\right\|$ 时，令 $p_1=\left(1+d/\left\|x_{12}\right\|\right)x_{12}$，显然有，

$$\begin{gathered}
\left\|x_{11}-p_1\right\|+\left\|x_{12}+p_1\right\|\ge d,\quad \left\|x_{11}-x_{12}-p_1\right\|+\left\|p_1\right\|\ge d,\quad \left\|x_{11}-x_{12}\right\|+\left\|2p_1\right\|\ge d,\\
\left\|x_{11}-2p_1\right\|+\left\|x_{12}\right\|\ge d,\quad \left\|x_{11}\right\|+\left\|x_{12}-2p_1\right\|\ge d,\quad \left\|x_{11}-p_1\right\|+\left\|x_{11}-x_{12}-p_1\right\|\ge d,\\
\left\|p_1\right\|+\left\|x_{12}+p_1\right\|\ge d,\quad \left\|x_{11}-2p_1\right\|+\left\|x_{11}-x_{12}\right\|\ge d,\quad \left\|x_{12}\right\|+\left\|2p_1\right\|\ge d,\\
\left\|x_{12}-p_1\right\|+\left\|2p_1\right\|\ge d,\quad \left\|x_{12}-p_1\right\|+\left\|p_1\right\|\ge d,\quad \left\|x_{12}-3p_1\right\|+\left\|p_1\right\|\ge d,\\
\left\|x_{12}-2p_1\right\|\ge d,\quad \left\|x_{12}\right\|+\left\|x_{12}-3p_1\right\|\ge d,\quad \left\|x_{12}-p_1\right\|+\left\|x_{12}-2p_1\right\|\ge d,\\
\left\|2p_1\right\|\ge d,\quad \left\|2p_1\right\|+\left\|p_1\right\|\ge d。
\end{gathered} \tag{2.4}$$

记

$$D_1f\left(A_1,\cdots,A_n\right)=\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(A_1,x_{21}+(-1)^{i_2}x_{22},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right),$$

$$D_1f\left(B_1,\cdots,B_n\right)=\sum_{j_2,\cdots,j_n\in\{1,2\}} f\left(B_1,x_{2j_2},\cdots,x_{nj_n}\right),$$

则

$$\begin{aligned}
&\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\\
&=D_1f\left(x_{11}+x_{12},A_2,\cdots,A_n\right)+D_1f\left(x_{11}-x_{12}-2p_1,A_2,\cdots,A_n\right)\\
&\quad-2^nD_1f\left(x_{11}-p_1,B_2,\cdots,B_n\right)-2^nD_1f\left(x_{12}+p_1,B_2,\cdots,B_n\right)\\
&\quad+D_1f\left(x_{11}-x_{12},A_2,\cdots,A_n\right)+D_1f\left(x_{11}-x_{12}-2p_1,A_2,\cdots,A_n\right)\\
&\quad-2^nD_1f\left(x_{11}-x_{12}-p_1,B_2,\cdots,B_n\right)-2^nD_1f\left(p_1,B_2,\cdots,B_n\right)\\
&\quad-\left[D_1f\left(x_{11}-x_{12}-2p_1,A_2,\cdots,A_n\right)+D_1f\left(x_{11}-x_{12}+2p_1,A_2,\cdots,A_n\right)\right.
\end{aligned}$$

$$-2^n D_1 f\left(x_{11}-x_{12},B_2,\cdots,B_n\right)-2^n D_1 f\left(2p_1,B_2,\cdots,B_n\right)\Big]$$
$$-\Big[D_1 f\left(x_{11}-x_{12}-2p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{11}+x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12},B_2,\cdots,B_n\right)-2^n D_1 f\left(x_{11}-2p_1,B_2,\cdots,B_n\right)\Big]$$
$$+D_1 f\left(x_{11}-x_{12}+2p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{11}+x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{11},B_2,\cdots,B_n\right)-2^n D_1 f\left(x_{12}-2p_1,B_2,\cdots,B_n\right)$$
$$-\Big[D_1 f\left(2x_{11}-x_{12}-2p_1,A_2,\cdots,A_n\right)+D_1 f((x_{12},A_2,\cdots,A_n)$$
$$-2^n D_1 f\left(x_{11}-p_1,B_2,\cdots,B_n\right)-2^n D_1 f((x_{11}-x_{12}-p_1,B_2,\cdots,B_n)$$
$$-\Big[D_1 f\left(x_{12}+2p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{12},A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12}+p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(p_1,B_2,\cdots,B_n\right)$$
$$+D_1 f\left(2x_{11}-x_{12}-2p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{11}-2p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(x_{11}-x_{12},B_2,\cdots,B_n\right)$$
$$+D_1 f\left(x_{12}+2p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12},B_2,\cdots,B_n\right)-2^n D_1 f\left(2p_1,B_2,\cdots,B_n\right)$$
$$+D_1 f\left(x_{12},A_2,\cdots,A_n\right)+D_1 f\left(x_{12}-4p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12}-2p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(2p_1,B_2,\cdots,B_n\right)$$
$$+D_1 f\left(x_{12},A_2,\cdots,A_n\right)+D_1 f\left(x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12}-p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(p_1,B_2,\cdots,B_n\right)$$
$$-\Big[D_1 f\left(x_{12}-4p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12}-3p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(p_1,B_2,\cdots,B_n\right)\Big]$$
$$-\Big[Df\left(x_{12}-2p_1,A_2,\cdots,A_n\right)+D_1 f\left(x_{12}-2p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12}-p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(0,B_2,\cdots,B_n\right)\Big]$$
$$+D_1 f\left(2x_{12}-3p_1,A_2,\cdots,A_n\right)+D_1 f\left(3p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12},B_2,\cdots,B_n\right)-2^n D_1 f\left(x_{12}-3p_1,B_2,\cdots,B_n\right)$$
$$-\Big[D_1 f\left(2x_{12}-3p_1,A_2,\cdots,A_n\right)+D_1 f\left(p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(x_{12}-p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(x_{12}-2p_1,B_2,\cdots,B_n\right)\Big]$$
$$-\Big[D_1 f\left(3p_1,A_2,\cdots,A_n\right)+D_1 f\left(p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(2p_1,B_2,\cdots,B_n\right)-2^n D_1 f\left(p_1,B_2,\cdots,B_n\right)\Big]$$
$$+\Big[D_1 f\left(p_1,A_2,\cdots,A_n\right)+D_1 f\left(p_1,A_2,\cdots,A_n\right)$$
$$-2^n D_1 f\left(0,B_2,\cdots,B_n\right)-2^n D_1 f\left(p_1,B_2,\cdots,B_n\right)\Big]$$

由(2.2)，(2.4)式及上面的关系可得

$$\left\|\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\right\|\le 17\delta \tag{2.5}$$

其中，$\forall x_{11}$，$x_{12}\in X$，$\|x_{i1}\|+\|x_{i2}\|\ge d$，$i\in\{2,\cdots,n\}$。现在设

$$\left\|\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\right\|\le 17^{n-1}\delta \tag{2.6}$$

其中，$\forall x_{i1}$，$x_{i2}\in X$，$i\in\{1,\cdots,n-1\}$，$\|x_{n1}\|+\|x_{n2}\|\ge d$。

$\|x_{n1}\|+\|x_{n2}\|<d$，如果 $x_{n1}=x_{n2}=0$，取 $p_n\in X$，$\|p_n\|=d$。否则，当 $\|x_{n1}\|\ge\|x_{n2}\|$ 时，令 $p_n=\left(1+d/\|x_{n1}\|\right)x_{n1}$，当 $\|x_{n1}\|<\|x_{n2}\|$ 时 $p_n=\left(1+d/\|x_{n2}\|\right)x_{n2}$。显然有，

$$\|x_{n1}-p_n\|+\|x_{n2}+p_n\|\ge d,\ \|x_{n1}-x_{n2}-p_n\|+\|p_n\|\ge d,\ \|x_{n1}-x_{n2}\|+\|2p_n\|\ge d,\ \|x_{n1}-2p_n\|+\|x_{n2}\|\ge d,$$
$$\|x_{n1}\|+\|x_{n2}-2p_n\|\ge d,\ \|x_{n1}-p_n\|+\|x_{n1}-x_{n2}-p_n\|\ge d,\ \|p_n\|+\|x_{n2}+p_n\|\ge d,\ \|x_{n1}-2p_n\|+\|x_{n1}-x_{n2}\|\ge d,$$
$$\|x_{n2}\|+\|2p_n\|\ge d,\ \|x_{n2}-p_n\|+\|2p_n\|\ge d,\ \|x_{n2}-p_n\|+\|p_n\|\ge d,\ \|x_{n2}-3p_n\|+\|p_n\|\ge d,\ \|x_{n2}-2p_n\|\ge d,$$
$$\|x_{n2}\|+\|x_{n2}-3p_n\|\ge d,\ \|x_{n2}-p_n\|+\|x_{n2}-2p_n\|\ge d,\ \|2p_n\|\ge d,\ \|2p_n\|+\|p_n\|\ge d \tag{2.7}$$

记

$$D_nf\left(A_1,\cdots,A_n\right)=\sum_{i_1,\cdots,i_n\in\{0,1\}}f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n-1,1}+(-1)^{i_{n-1}}x_{n-1,2},A_n\right),$$

$$D_nf\left(B_1,\cdots,B_n\right)=\sum_{j_1,\cdots,j_n\in\{1,2\}}f\left(x_{1j_1},\cdots,x_{n-1,j_{n-1}},B_n\right)。$$

则

$$\begin{aligned}
&\sum_{i_1,\cdots,i_n\in\{0,1\}}f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}}f\left(x_{1j_1},\cdots,x_{nj_n}\right)\\
&=D_nf\left(A_1,\cdots,A_{n-1},x_{n1}+x_{n2}\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}-2p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-p_n\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}+p_n\right)\\
&\quad+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}-2p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-x_{n2}-p_n\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},p_n\right)\\
&\quad-\left[D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}-2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}+2p_n\right)\right.\\
&\quad\left.-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-x_{n2}\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},2p_n\right)\right]\\
&\quad-\left[D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}-2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}+x_{n2}-2p_n\right)\right.\\
&\quad\left.-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-2p_n\right)\right]\\
&\quad+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}-x_{n2}+2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n1}+x_{n2}-2p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}-2p_n\right)\\
&\quad-\left[D_nf\left(A_1,\cdots,A_{n-1},2x_{n1}-x_{n2}-2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}\right)\right.\\
&\quad\left.-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-p_n\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-x_{n2}-p_n\right)\right]\\
&\quad-\left[D_nf\left(A_1,\cdots,A_{n-1},x_{n2}+2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}\right)\right.\\
&\quad\left.-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}+p_n\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},p_n\right)\right]\\
&\quad+D_nf\left(A_1,\cdots,A_{n-1},2x_{n1}-x_{n2}-2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}-2p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-2p_n\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n1}-x_{n2}\right)\\
&\quad+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}+2p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}-2p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},2p_n\right)\\
&\quad+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}-4p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}-2p_1\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},2p_n\right)\\
&\quad+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}-2p_n\right)\\
&\quad-2^nD_nf\left(B_1,\cdots,B_{n-1},x_{n2}-p_1\right)-2^nD_nf\left(B_1,\cdots,B_{n-1},p_n\right)\\
&\quad-\left[D_nf\left(A_1,\cdots,A_{n-1},x_{n2}-4p_n\right)+D_nf\left(A_1,\cdots,A_{n-1},x_{n2}-2p_n\right)\right.
\end{aligned}$$

$$-2^n D_n f(B_1,\cdots,B_{n-1},x_{n2}-3p_1)-2^n D_n f((B_1,\cdots,B_{n-1},p_n)]$$
$$-[D_n f(A_1,\cdots,A_{n-1},x_{n2}-2p_n)+D_n f(A_1,\cdots,A_{n-1},x_{n2}-2p_n)$$
$$-2^n D_n f(B_1,\cdots,B_{n-1},x_{n2}-2p_1)-2^n D_n f(B_1,\cdots,B_{n-1},0)]$$
$$+D_n f(A_1,\cdots,A_{n-1},2x_{n2}-3p_n)+D_n f(A_1,\cdots,A_{n-1},3p_n)$$
$$-2^n D_n f(B_1,\cdots,B_{n-1},x_{n2})-2^n D_n f(B_1,\cdots,B_{n-1},x_{n2}-3p_n)$$
$$-[D_n f(A_1,\cdots,A_{n-1},2x_{n2}-3p_n)+D_n f(A_1,\cdots,A_{n-1},p_n)$$
$$-2^n D_n f(B_1,\cdots,B_{n-1},x_{n2}-p_n)-2^n D_n f(B_1,\cdots,B_{n-1},x_{n2}-2p_n)]$$
$$-[D_n f(A_1,\cdots,A_{n-1},3p_n)+D_n f(A_1,\cdots,A_{n-1},p_n)$$
$$-2^n D_n f(B_1,\cdots,B_{n-1},2p_n)-2^n D_n f(B_1,\cdots,B_{n-1},p_n)]$$
$$+D_n f(A_1,\cdots,A_{n-1},p_n)+D_n f(A_1,\cdots,A_{n-1},p_n)$$
$$-2^n D_n f(B_1,\cdots,B_{n-1},p_n)-2^n D_n f(B_1,\cdots,B_{n-1},0)$$

由(2.6)，(2.7)式及上面的关系式可得

$$\left\|\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(x_{11}+(-1)^{i_1}x_{12},\cdots,x_{n1}+(-1)^{i_n}x_{n2}\right)-2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(x_{1j_1},\cdots,x_{nj_n}\right)\right\|\le 17^n\delta \tag{2.8}$$

其中，$\forall x_{11},x_{12},\cdots,x_{n1},x_{n2}\in X$ 。

II. 证明(2.3)式成立。

令(2.8)中 $x_{i1}=x_{i2}=x_i$， $i\in\{1,\cdots,n\}$，然后再除 4^n 。由已知，当 $(x_1,\cdots,x_n)\in X^n$ 至少有一个元素为 0，$f(x_1,\cdots,x_n)=0$，可得对 $\forall x_1,\cdots,x_n\in X$，

$$\left\|\frac{1}{4^n}f(2x_1,\cdots,2x_n)-f(x_1,\cdots,x_n)\right\|\le\frac{17^n}{4^n}\delta 。$$

用 $2^k x_i$ 代替 x_i， $i\in\{1,\cdots,n\}$，不等式两端同除 4^{nk} 可得对 $\forall x_1,\cdots,x_n\in X$，

$$\left\|\frac{1}{4^{n(k+1)}}f\left(2^{k+1}x_1,\cdots,2^{k+1}x_n\right)-\frac{1}{4^{nk}}f\left(2^k x_1,\cdots,2^k x_n\right)\right\|\le\frac{17^n}{4^{n(k+1)}}\delta 。$$

因此对于非负整数 $m>k\ge 0$， $\forall x_1,\cdots,x_n\in X$， 有

$$\left\|\frac{1}{4^{nm}}f\left(2^m x_1,\cdots,2^m x_n\right)-\frac{1}{4^{nk}}f\left(2^k x_1,\cdots,2^k x_n\right)\right\|\le\sum_{i=k}^{m-1}\frac{17^n}{4^{n(i+1)}}\delta \tag{2.9}$$

令(2.9)式中 $k\to\infty$ 可得 $\left\{\frac{1}{4^{nk}}f\left(2^k x_1,\cdots,2^k x_n\right)\right\}_{k\in N}$ 是 Y 中的 Cauchy 列，由于 Y 是 Banach 空间，所以 Cauchy 列收敛。记

$$F(x_1,\cdots,x_n)=\lim_{k\to\infty}\frac{1}{4^{nk}}f\left(2^k x_1,\cdots,2^k x_n\right),\quad \forall x_1,\cdots,x_n\in X \tag{2.10}$$

令(2.9)式中 $k=0$，

$$\left\|\frac{1}{4^{nm}}f\left(2^m x_1,\cdots,2^m x_n\right)-f(x_1,\cdots,x_n)\right\|\le\sum_{i=0}^{m-1}\frac{17^n}{4^{n(i+1)}}\delta$$

再令 $m\to\infty$， 由(2.10)式可得，对 $\forall x_1,\cdots,x_n\in X$，

$$\left\|F\left(x_1,\cdots,x_n\right)-f\left(x_1,\cdots,x_n\right)\right\|\le\frac{17^n}{4^n-1}\delta$$

III. 证明函数 F 是多元二次的。

将(2.8)式中用 $2^k x_{i1}$， $2^k x_{i2}$ 分别代替 x_{i1}， x_{i2}， $i\in\{1,\cdots,n\}$，不等式两端同除 4^{nk}，可得

$$\begin{aligned}&\left\|\frac{1}{4^{nk}}\sum_{i_1,\cdots,i_n\in\{0,1\}} f\left(2^k x_{11}+(-1)^{i_1}2^k x_{12},\cdots,2^k x_{n1}+(-1)^{i_n}2^k x_{n2}\right)\right.\\&\left.-\frac{1}{4^{nk}}2^n\sum_{j_1,\cdots,j_n\in\{1,2\}} f\left(2^k x_{1j_1},\cdots,2^k x_{nj_n}\right)\right\|\le\frac{1}{4^{nk}}17^n\delta\end{aligned}\tag{2.11}$$

其中 $\forall x_{11},x_{12},\cdots,x_{n1},x_{n2}\in X$ 。令 $k\to\infty$ 得 F 是多元二次的。

IV. 证明函数 F 是唯一的。

假设有 F' 满足(2.3)式，由 F' 和 F 的多元二次性可得，

$$\begin{aligned}\left\|F\left(x_1,\cdots,x_n\right)-F'\left(x_1,\cdots,x_n\right)\right\|&\le\frac{1}{4^{nk}}\left\|F\left(2^k x_1,\cdots,2^k x_n\right)-F'\left(2^k x_1,\cdots,2^k x_n\right)\right\|\\&\le\frac{1}{4^{nk}}\left\|F\left(2^k x_1,\cdots,2^k x_n\right)-f\left(2^k x_1,\cdots,2^k x_n\right)\right\|\\&\quad+\frac{1}{4^{nk}}\left\|f\left(2^k x_1,\cdots,2^k x_n\right)-F'\left(2^k x_1,\cdots,2^k x_n\right)\right\|\\&\le\frac{2}{4^{nk}}\frac{17^n}{4^n-1}\delta\end{aligned},$$

令 $k\to\infty$ 可得 $F=F'$，从而 $F:X^n\to Y$ 是唯一的多元二次函数满足(2.3)式。

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